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Computation of the local generalized H-Lefschetz number ☆

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Abstract

Let X be a connected, locally finite polyhedron. For U a compact, connected subpolyhedron of X, let $f:U\to X$ be a map with $\mathrm{Fix}(f)\cap\partial U=\emptyset$. The local H-Nielsen number $N(f)$ is a lower bound for the number of fixed points of any map in a restricted homotopy class of f. We prove that the local generalized H-Lefschetz number $L_H(f;\, \tilde{f},\, \tilde{\imath})$ can, in certain cases, be expressed in terms of the isotropy subgroups of the local Reidemeister action and the local Lefschetz numbers of the lifts of f. From $L_H(f;\, \tilde{f},\, \tilde{\imath})$ we determine $N(f)$. We consider applications to lens spaces and to a spherical space form.

Keywords: Local Nielsen number; Lefschetz number; Generalized Lefschetz number; Topological fixed point theory; Nielsen fixed point theory

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1. Introduction

The local generalized H-Lefschetz number, $L_H(f;\, \tilde{f},\, \tilde{\imath})$, is defined in [5] as the alternating sum of trace-like elements from a free Abelian $\mathbb{Z}$-module. Under certain hypotheses, $L_H(f;\, \tilde{f},\, \tilde{\imath})$ provides information about the local H-Nielsen number. The definition of $L_H(f;\, \tilde{f},\, \tilde{\imath})$ does not provide a method for calculating its value. We prove that, under certain conditions, the local generalized H-Lefschetz number can be expressed in terms of computable information.

Let X be a connected, finite-dimensional, locally compact polyhedron. Let $f:U\to X$ be a map defined on an open, connected subset U of X. The fixed point set of f is $\mathrm{Fix}\, f=\{x\in U: f(x)=x\}$, and we consider only maps f with Fix f

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compact. The local H-Nielsen number of f is a lower bound for $|\text{Fix } g|$ for all $g \simeq f$ via an admissible homotopy. References for Nielsen theory (with $U = X$) are [1,10,11]. Local Nielsen theory is introduced in [3]. We prove that, when the lift of f is a compact map and all isotropy subgroups of the Reidemeister action are finite, the local generalized H-Lefschetz number of f can be expressed in terms of the isotropy subgroups and a lift index. This lift index, defined in the next section, is a generalization of the local Lefschetz number as in [6]. We define a local weak Jiang setting. For such a setting, the lift index reduces to the local Lefschetz number. These results are a generalization of results by Fadell and Husseini in [2].

Section 4 contains two examples for which the local generalized H-Lefschetz number can be calculated. In the first, X is a spherical space form with $U = X$. For any self-map of X, the resulting setting is a weak Jiang setting, while X itself is not a Jiang space. The second example involves calculating local Nielsen numbers for maps from a solid torus into a lens space.

The background necessary for the results in [5] and for those proven here is provided in the expository paper [7].

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2. Preliminaries

Let X be a connected, finite-dimensional, locally compact polyhedron. For U an open, connected subset of X, we define $\partial U = \overline{U} \cap \overline{X - U}$. Let $f : U \to X$ be a map with $\text{Fix } f = \{x \in U\colon f(x) = x\}$, the set of fixed points of f. We consider only maps for which $\text{Fix } f$ is compact, and we say f is compactly fixed. Let H be a normal subgroup of $\pi_1(X)$. Local H-Nielsen theory involves partitioning the fixed points of f into equivalence classes. (See [8,13].) Two fixed points, x and y, are in the same equivalence class if and only if there exists a path ω in U from x to y for which the loop $(f \circ \omega) * \omega^{-1}$ is in a loop class in H. The equivalence classes of fixed points are called local H-Nielsen classes for f. Each class is assigned an integer called the local index of the class. (See [6].)

A class of fixed points of f with nonzero index is called essential, because it cannot be removed by a deformation of f without introducing new fixed points. A homotopy $h : U \times I \to X$ is admissible if $\bigcup_{t \in I} \text{Fix } h_t$ is a compact subset of U. It is well known that there is a one-to-one correspondence between the essential classes of f and the essential classes of g, whenever $g \simeq f$ via an admissible homotopy. The local H-Nielsen number of f, denoted by $N_H(f)$, is the number of essential local H-Nielsen classes. Thus $N_H(f)$ is invariant under admissible homotopy. We have $N_H(f) \leqslant \min\{|\text{Fix } g|\colon g \simeq f\}$ where the homotopies must be admissible.

A subset K of U is an (f, U)-subset of X if it is a compact, connected subset of U with $\text{Fix } f \subseteq \text{int } K$. The equivalence classes for f on K (where the path ω must be in K) are local $(K; H)$-Nielsen classes for f. It is possible to choose K large enough so that each local $(K; H)$-Nielsen class equals a local H-Nielsen class for f on U. Such a K is called stable. For K stable, $N_H(f)$ can be calculated on K rather than on all of U. It is also possible to choose $K = \overline{\operatorname{int} K}$ without loss of information. See [5]. We assume from now on that K is a stable (f, U)-subset of X with $K = \overline{\operatorname{int} K}$.

Let $\tilde{X}$ be the regular covering space for X for which the group of covering transformations, $\tilde{\pi}_X$, is isomorphic to $\pi_1(X)/H$. Let $\tilde{p}_X : \tilde{X} \to X$ denote the covering projection, and let $i : K \hookrightarrow X$ be the inclusion map.

A regular covering space $\tilde{K}$ of K with covering projection $\tilde{p}_K$ is (H, f)-admissible if there exist maps $\tilde{\imath}$ and $\tilde{f}$ for which $\tilde{p}_X \tilde{f} = f \tilde{p}_K$ and $\tilde{p}_X \tilde{\imath} = i \tilde{p}_K$. The maps $\tilde{f}$ and $\tilde{\imath}$ are said to be lifts of f and i, respectively. The universal cover of K is always (H, f)-admissible. Thus the set of (H, f)-admissible covers for a given f and a given H is always nonempty.

Definition 2.1 (*Local setting for f*). Let $f : U \to X$ be a map with Fix f compact. A local setting LS(f) for f consists of $\{X, K, f, \tilde{X}, \tilde{K}, \tilde{f}, \tilde{\imath}\}$ as above, with $K = \overline{\operatorname{int} K}$ a stable (f, U)-subset of X and $\tilde{K}$ an (H, f)-admissible covering space for K.

Let $\tilde{\pi}_K$ be the group of covering transformations for $\tilde{p}_K$. Each lift of f to $\tilde{K}$ may be written uniquely as $\alpha \tilde{f}$ for some $\alpha \in \tilde{\pi}_X$. An analogous statement is true for $\tilde{\imath}$. The map $\tilde{f}$ induces a homomorphism $\tilde{\phi} : \tilde{\pi}_K \to \tilde{\pi}_X$ given by $\tilde{f}(\tau \tilde{y}) = \tilde{\phi}(\tau) \tilde{f}(\tilde{y})$ for all $\tau \in \tilde{\pi}_K$ and all $\tilde{y} \in \tilde{K}$. The lift $\tilde{\imath}$ of the inclusion $i : K \hookrightarrow X$ induces a homomorphism $\tilde{\xi} : \tilde{\pi}_K \to \tilde{\pi}_X$. As expected, for all $\tau \in \tilde{\pi}_K$ and $\tilde{y} \in \tilde{K}$, $\tilde{\imath}(\tau \tilde{y}) = \tilde{\xi}(\tau) \tilde{\imath}(\tilde{y})$.

The Reidemeister action of $\tilde{\pi}_K$ on $\tilde{\pi}_X$ is given by the following group action. For any $\tau \in \tilde{\pi}_K$ and any $\alpha \in \tilde{\pi}_X$, $\tau \cdot \alpha = \tilde{\xi}(\tau) \alpha \tilde{\phi}(\tau^{-1})$. Let $[\alpha]$ denote the orbit of α under the Reidemeister action, and let $R(\tilde{\phi}, \tilde{\xi})$ denote the set of Reidemeister orbits. Let $\alpha \in \tilde{\pi}_X$ and define $\operatorname{Coin}(\alpha \tilde{f}, \tilde{\imath}) := \{\tilde{x} \in \tilde{K} : \alpha \tilde{f}(\tilde{x}) = \tilde{\imath}(\tilde{x})\}$. As in [5], the set $\tilde{p}_K(\operatorname{Coin}(\alpha \tilde{f}, \tilde{\imath}))$ either is a local H-Nielsen class for f or is the empty set. Let β be in $\tilde{\pi}_X$. If $\operatorname{Coin}(\alpha \tilde{f}, \tilde{\imath}) \neq \emptyset$ and $\operatorname{Coin}(\beta \tilde{f}, \tilde{\imath}) \neq \emptyset$, we have $\tilde{p}_K(\operatorname{Coin}(\alpha \tilde{f}, \tilde{\imath}))$ equal to $\tilde{p}_K(\operatorname{Coin}(\beta \tilde{f}, \tilde{\imath}))$ if and only if $[\alpha] = [\beta]$. A set of Reidemeister representatives is a subset of $\tilde{\pi}_X$ containing exactly one element of each orbit in $R(\tilde{\phi}, \tilde{\xi})$. If $\operatorname{Coin}(\alpha \tilde{f}, \tilde{\imath}) \neq \emptyset$, let N_H^α denote the local H-Nielsen class $\tilde{p}_K \operatorname{Coin}(\alpha \tilde{f}, \tilde{\imath})$.

As in [5], we use the following notation for a formal sum that is a well-known part of Nielsen fixed point theory.

Definition 2.2 (*The $(H, K, \tilde{f}, \tilde{\imath})$-NR chain for f*). Let W be a set of Reidemeister representatives, and let K be a stable (f, U)-subset of X.

The $(H, K, \tilde{f}, \tilde{\imath})$-NR (Nielsen–Reidemeister) chain for f is

$$N_H(f; \tilde{f}, \tilde{\imath}) = \sum_{\alpha \in W} i(N_H^\alpha)[\alpha].$$

Here $i(N_H^\alpha)$ is the local index of the H-Nielsen class N_H^α.

The $(H, K, \tilde{f}, \tilde{\imath})$-NR chain is independent of K as long as K is a stable (f, U)-subset of X.

2.1. The definition of $L_H(f; \tilde{f}, \tilde{\imath})$

We identify X with a triangulation of X. For L any simplicial complex that is a subdivision of X, a subcomplex K of L is an (f, U)-subcomplex if the underlying space of K is an (f, U)-subset of X. We do not distinguish between a simplicial complex and its underlying space, and all simplices are assumed to be oriented. We study only those subdivisions L of X for which there is a stable (f, U)-subcomplex. Let $\mathrm{LS}(f) = \{L, K, f, \tilde{L}, \tilde{K}, \tilde{f}, \tilde{\imath}\}$ be a local setting for which K is a stable (f, U)-subcomplex of L. The covering spaces $\tilde{L}$ and $\tilde{K}$ inherit simplicial structures from L and K. For K an (f, U)-subset of X, a homotopy $h: K \times I \to X$ is admissible if $\bigcup_{t \in I} \mathrm{Fix}\ h_t$ is compact in the interior of K.

We would like to have a simplicial approximation to f with no fixed points on ∂K that is homotopic to f via an admissible homotopy. As in [6], there exist subdivisions L' of L, K' of K and K'' of K', with K' a subcomplex of L', so that there is a simplicial approximation to f, $g: K'' \to L'$, with the desired properties. Let $\tilde{K}''$ and $\tilde{L}'$ be the obvious subdivisions of $\tilde{K}$ and $\tilde{L}$, respectively. We orient the simplices of $\tilde{K}''$ and $\tilde{L}'$ so that $\tilde{p}_K$ and $\tilde{p}_X$ preserve orientation.

Let $\tilde{J}$ be the lift of the admissible homotopy between f and g with $\tilde{J}_0 = \tilde{f}$, and define $\tilde{g} = \tilde{J}_1$. Let $C_q(\cdot; \mathbb{Z})$ be the group of oriented simplicial chains, and let $\overline{K} = \tilde{p}_X^{-1}(K)$. Let τ_q be the standard subdivision chain map, and let pr_q be the projection function. We define Φ_q to be the composition

$$\Phi_q : C_q(\tilde{K}'; \mathbb{Z}) \xrightarrow{\tau_q} C_q(\tilde{K}''; \mathbb{Z}) \xrightarrow{C_q(\tilde{g})} C_q(\tilde{L}'; \mathbb{Z}) \xrightarrow{\mathrm{pr}_q} C_q(\overline{K}'; \mathbb{Z}).$$

Let $M_q = [m_{i,j}]$ be the square matrix for Φ_q over $\mathbb{Z}[\tilde{\pi}_X]$. For $\mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$ the free Abelian $\mathbb{Z}$-module generated by the set $R(\tilde{\phi}, \tilde{\xi})$, let $\rho: \mathbb{Z}[\tilde{\pi}_X] \to \mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$ be the linear function defined by $\rho: \theta \mapsto [\theta^{-1}]$ for each $\theta \in \tilde{\pi}_X$.

We define $T^R(\Phi_q)$ to be a trace-like element of $\mathbb{Z}(R(\tilde{\phi}, \tilde{\xi}))$ given by

$$T^R(\Phi_q) = \rho \circ \mathrm{tr}(M_q) \in \mathbb{Z}\big(R(\tilde{\phi}, \tilde{\xi})\big).$$

Definition 2.3 (*The local generalized H-Lefschetz number*). The local generalized H-Lefschetz number, $L_H(f; \tilde{f}, \tilde{\imath})$, is defined to be

$$L_H(f; \tilde{f}, \tilde{\imath}) = \sum_q (-1)^q T^R(\Phi_q).$$

Note that ρ involves a twisting in the sense that we write $[\theta^{-1}]$ where one might expect to see $[\theta]$. This notation corresponds to the work in [2,5]. In [5], it is shown that $L_H(f; \tilde{f}, \tilde{\imath})$ is independent of the choices made in its definition as long as K is a stable subset of X. It is also shown that $L_H(f; \tilde{f}, \tilde{\imath})$ is invariant under homotopy and has an additivity property.

2.2. The lift index

Let LS(f) be a local setting as in Definition 2.1 for which $\tilde{f}$ is compact. We define the lift index $\Lambda(\tilde{f})$ of $\tilde{f}$ on $\tilde{K}$. $\Lambda(\tilde{f})$ is similar to $L_H(f;\ \tilde{f},\ \tilde{\imath})$ in that it is an alternating sum. But the lift index is an integer, while $L_H(f;\ \tilde{f},\ \tilde{\imath})$ is an element of $\mathbb{Z}(R(\tilde{\phi},\ \tilde{\xi}))$. For the lift index, the projection used is more restrictive than for $L_H(f;\ \tilde{f},\ \tilde{\imath})$. $\Lambda(\tilde{f})$ is an extension of the local simplicial index of $\tilde{f}$ as in [6]. If $\tilde{K}$ is a component of $\tilde{p}_L^{-1}(K)$, the lift index reduces to the local simplicial index.

Note that $\tilde{\imath}(\tilde{K}')$ is a subcomplex of, but is not in general equal to, $\overline{K}'$. Let $\pi : C(\overline{K}';\ \mathbb{Z}) \to C(\tilde{\imath}(\tilde{K}');\ \mathbb{Z})$ be the projection map, and define $\tilde{G}_q$ to be the composition $\tilde{G}_q = \pi_q \Phi_q$.

For any simplicial complex R, let R_q be the set of positively oriented q-simplices of R. As in [6], for each $s \in \tilde{K}'_q$, let $s_{\tilde{K}''}$ be the subdivision of s that is a subcomplex of $\tilde{K}''$. Let $\mathscr{O}(s)$ be the set of positively oriented q-simplices in $s_{\tilde{K}''}$. It is well known that for any q-simplex m of $\tilde{K}'$, $\tau_q(m) = \Sigma_{r \in \mathscr{O}(m)} r$. We have, for m any q-simplex of $\tilde{K}'$, $C_q(\tilde{g})\tau_q(m) = \Sigma_{r \in \mathscr{O}(m)} \tilde{g}(r) = \Sigma_{r \in \mathscr{O}(m)} \Sigma_{s \in \tilde{L}'_q} \lambda_{r,s} s \in C_q(\tilde{L}';\ \mathbb{Z})$ with $\lambda_{r,s} \in \{0,\ 1,\ -1\}$. Thus $\tilde{G}_q(m) = \Sigma_{r \in \mathscr{O}(m)} \Sigma_{s \in \tilde{L}'_q} \lambda_{r,s} \pi_q \mathrm{pr}_q(s) \in C_q(\tilde{\imath}(\tilde{K}');\ \mathbb{Z})$.

The following fact will be useful in the proof of Theorem 3.2.

Proposition 2.4. *For any $\sigma \in \tilde{\pi}_K$ and any $r \in \mathscr{O}(m)$ with m a q-simplex of $\tilde{K}'$, and for any $s \in \tilde{L}'_q$, we have $\lambda_{\sigma r,s} = \lambda_{r,\tilde{\phi}(\sigma^{-1})s}$. If $\sigma \in \ker \tilde{\xi}$, then $\lambda_{\sigma r,\tilde{\imath}(\sigma m)} = \lambda_{\sigma r,\tilde{\imath}(m)} = \lambda_{r,\tilde{\phi}(\sigma^{-1})\tilde{\imath}(m)}$.*

Proof. We want $\lambda_{\sigma r,s}$, which is the coefficient of s in $\tilde{g}(\sigma r)$. Note that

$$\tilde{g}(\sigma r) = \tilde{\phi}(\sigma)\tilde{g}(r) = \tilde{\phi}(\sigma) \sum_{s \in \tilde{L}'_q} \lambda_{r,s} s = \sum_{s \in \tilde{L}'_q} \lambda_{r,s} \tilde{\phi}(\sigma) s.$$

The coefficient of s is $\lambda_{r,\tilde{\phi}(\sigma^{-1})s}$. $\square$

For any $y \in \tilde{L}'_q$ and for any $x \in \tilde{K}'_q$, let $\mu_{x,y} = \Sigma_{r \in \mathscr{O}(x)} \lambda_{r,y} \in \mathbb{Z}$. Let $T(\tilde{G}_q)$ be a trace-like integer for $\tilde{G}_q$ over $\mathbb{Z}$ given by

$$T\left(\tilde{G}_q\right) = \sum_{x \in \tilde{K}'_q} \mu_{x,\tilde{\imath}(x)} \in \mathbb{Z}.$$

Definition 2.5 (*The lift index for a simplicial map*). For $\tilde{g} : \tilde{K}'' \to \tilde{L}'$ a compact simplicial map, we define the lift index of $\tilde{g}$ on $\tilde{K}''$ to be

$$\Lambda(\tilde{g}) = \sum_q (-1)^q T\left(\tilde{G}_q\right) \in \mathbb{Z}.$$

We require $\tilde{g}$ to be compact in order to be sure that at most finitely many of the terms in the sum are nonzero.

Proposition 2.6. *The lift index of $\tilde{g}$ on $\tilde{K}''$ satisfies*

$$\Lambda(\tilde{g}) = \sum_q (-1)^q \sum_{v \in V_q} \sum_{r \in \mathscr{O}(v)} \sum_{\sigma \in \ker \xi} \lambda_{\sigma r, \tilde{\imath}(v)}$$

with V_q any set of q-simplices of $\tilde{K}'$ for which $\tilde{\imath}$ is a bijection of V_q onto the q-simplices of $\tilde{\imath}(\tilde{K}')$.

Note that $x \in \tilde{K}'_q$ is not necessarily in V_q, but $\tilde{\imath}(x)$ is an element of $\tilde{\imath}(V_q)$.

Proof. We have $\tilde{K}_q = \{\eta v: \eta \in \ker \xi, v \in V_q\}$, and $T(\tilde{G}_q) = \sum_{\eta \in \ker \xi} \sum_{v \in V_q} \mu_{\eta v, \tilde{\imath}(v)}$. Thus $T(\tilde{G}_q) = \sum_{\eta \in \ker \xi} \sum_{v \in V_q} \sum_{r \in \mathscr{O}(v)} \lambda_{\eta r, \tilde{\imath}(v)}$. □

Definition 2.7. (*The lift index for a local setting for f*). Given LS(f) a local setting for f with $\tilde{f}$ a compact map, let $\tilde{g}: \tilde{K}'' \to \tilde{L}'$ be a simplicial approximation to $\tilde{f}$. The lift index of $\tilde{f}$ on $\tilde{K}$ is defined to be $\Lambda(\tilde{f}) := \Lambda(\tilde{g})$.

Proposition 2.8. *The lift index of $\tilde{f}$ on $\tilde{K}$ is independent of the choice of $\tilde{g}$ whenever $\tilde{g}$ covers a simplicial approximation to f. The lift index of $\tilde{f}$ on $\tilde{K}$ is also independent of the choices of subdivisions $\tilde{K}''$ and $\tilde{L}'$.*

The proof is analogous to that of [6, Lemma 3.3].

In Theorem 3.2, we will break down $L_H(f;\, \tilde{f}, \tilde{\imath})$ into a sum with each Reidemeister orbit having a coefficient involving the lift index. The theorem holds under certain conditions of compactness and finiteness. We will define a local weak Jiang setting, where the lift index is independent of the lift chosen. When LS(f) is a local weak Jiang setting, the calculation of $L_H(f;\, \tilde{f}, \tilde{\imath})$ is simplified. These results will be used for examples in the last section.

2.3. Properties of the lift index

Proposition 2.9 (Invariance of the lift index under homotopy). *For any local setting* LS(f) *let $\tilde{h}: \tilde{K} \to \tilde{L}$ be a map that is homotopic to $\tilde{f}$ via a homotopy J. We require that J preserve fibers and that, for each $t \in I$,* $\mathrm{Coin}(J_t, \tilde{\imath}) \cap \partial\tilde{K} = \emptyset$. *Then $\Lambda(\tilde{f}) = \Lambda(\tilde{h})$.*

Proof. The proof of the invariance of the lift index under homotopy is analogous to the proof in [6] for the local simplicial index. Note that $\tilde{K}$ is not necessarily compact. We require J to preserve fibers so that we can make use of the compactness of K in the proof. Fournier's proof uses compactness. □

To calculate the local generalized H-Lefschetz number in Section 3, we must consider one lift of f for each Reidemeister orbit. The next proposition provides a connection between the lift index and local H-Nielsen classes of a map. It corresponds to [2, equation 6.25]. The proof involves a counting argument and a local version of a Hopf approximation to f. For a proof of the latter, see [4].

Proposition 2.10. *For any local setting* LS(f) *and any* $\alpha \in \tilde{\pi}_X$, *let* $(\tilde{\pi}_K)_\alpha$ *be the isotropy subgroup for* α *under the Reidemeister action. Then the lift index of* $\alpha\tilde{f}$ *equals the product of the size of* $(\tilde{\pi}_K)_\alpha$ *and the index of the local H-Nielsen class induced by* α. *That is,*

$$\Lambda(\alpha\tilde{f}) = i(N_H^\alpha)\,|(\tilde{\pi}_K)_\alpha|.$$

Proposition 2.11. *For any* $\rho \in \tilde{\pi}_K$ *and any* $\alpha \in \tilde{\pi}_X$, *we have* $\Lambda(\tilde{\xi}(\rho)\alpha\tilde{\phi}(\rho^{-1})\tilde{f}) = \Lambda(\alpha\tilde{f})$.

Proof. This follows directly from Proposition 2.10, because both the index of a local H-Nielsen class and the order of an isotropy subgroup are independent of the choice of representative of a Reidemeister class. Let $\beta = \tilde{\xi}(\rho)\alpha\tilde{\phi}(\rho^{-1})$. Then $[\alpha] = [\beta]$ and $N_H^\alpha = N_H^\beta$. There is a one-to-one correspondence between $(\tilde{\pi}_K)_\beta$ and $(\tilde{\pi}_K)_\alpha$ given by conjugation by ρ. Thus the lift index of $\beta\tilde{f}$ equals the lift index of $\alpha\tilde{f}$. $\square$

Let W be a set of Reidemeister representatives. Given a local setting LS(f) with $\tilde{f}$ compact and with $(\tilde{\pi}_K)_\alpha$ finite for every $\alpha \in \tilde{\pi}_X$, let the amalgamated lift index for LS(f) be

$$A_H(f;\ \tilde{f}, \tilde{\imath}) = \sum_{\alpha \in W} \frac{\Lambda(\alpha\tilde{f})}{|(\tilde{\pi}_K)_\alpha|}[\alpha].$$

Note that $A_H(f;\ \tilde{f}, \tilde{\imath})$ is defined in terms of one lift for each Reidemeister orbit, while $L_H(f;\ \tilde{f}, \tilde{\imath})$ is defined in terms of a single lift.

3. The local generalized H-Lefschetz number and NR chains

Let LS(f) be a local setting for f with $\tilde{f}$ compact. The results of this section are generalizations of work by Fadell and Husseini in [2], where the special case $U = X$ is considered. We will prove that $L_H(f;\ \tilde{f}, \tilde{\imath}) = N_H(f;\ \tilde{f}, \tilde{\imath}) = A_H(f;\ \tilde{f}, \tilde{\imath})$ when certain conditions of finiteness are met. At the end of the section, there is a discussion of these conditions of finiteness.

3.1. The function T_α^R

Let LS(f) be a local setting for f with $\tilde{f}$ a compact map. Let $\tilde{g} : \tilde{K}'' \to \tilde{L}'$ be a simplicial approximation to $\tilde{f}$ as above. Let $B_q \subseteq \tilde{K}'$ be a set consisting of one lift to $\tilde{K}'$ of each q-simplex in K'. For any $\alpha \in \tilde{\pi}_X$, $C_q(\tilde{K}';\ \mathbb{Q})$ is a $\mathbb{Q}[(\tilde{\pi}_K)_\alpha]$-module with basis B_q, and $C_q(\tilde{\imath}(\tilde{K}');\ \mathbb{Q})$ is a $\mathbb{Q}[\tilde{\xi}(\tilde{\pi}_K)_\alpha]$-module with basis $\tilde{\imath}(B_q)$.

For any $\alpha \in \tilde{\pi}_X$, we define the homomorphism $\alpha\tilde{G}_q : C_q(\tilde{K}'; \mathbb{Q}) \to C_q(\tilde{\imath}(\tilde{K}'); \mathbb{Q})$ to be $\alpha\tilde{G}_q = \pi_q \mathrm{pr}_q C_q(\alpha\tilde{g})\tau_q$.

Proposition 3.1. *For any $\alpha \in \tilde{\pi}_X$, the homomorphism $\alpha\tilde{G}_q : C_q(\tilde{K}'; \mathbb{Q}) \to C_q(\tilde{\imath}(\tilde{K}'); \mathbb{Q})$ respects the module structures of $C_q(\tilde{K}'; \mathbb{Q})$ and $C_q(\tilde{\imath}(\tilde{K}'); \mathbb{Q})$. In other words, for any $\sigma \in (\tilde{\pi}_K)_\alpha$ and any t a q-simplex of $\tilde{K}'$, we have $\alpha\tilde{G}_q(\sigma t) = \tilde{\xi}(\sigma)\alpha\tilde{G}_q(t)$.*

Proof. We have $\alpha\tilde{G}_q(\sigma t) = \pi_q \mathrm{pr}_q C_q(\alpha\tilde{g})\tau_q(\sigma t) = \pi_q \mathrm{pr}_q C_q(\alpha\tilde{\phi}(\sigma)\tilde{g})\tau_q(t)$. Note that $\sigma \in (\tilde{\pi}_K)_\alpha$ implies that $\tilde{\xi}(\sigma)\alpha = \alpha\tilde{\phi}(\sigma)$. Thus $\alpha\tilde{G}_q(\sigma t) = \pi_q \mathrm{pr}_q C_q(\tilde{\xi}(\sigma)\alpha\tilde{g})\tau_q(t)$. We have $\pi_q \mathrm{pr}_q(\tilde{\xi}(\sigma)s) = \tilde{\xi}(\sigma)\pi_q \mathrm{pr}_q(s)$ for all $s \in \tilde{L}'_q$. Thus $\alpha\tilde{G}_q(\sigma t) = \tilde{\xi}(\sigma)\alpha\tilde{G}_q(t)$. □

We restrict our study to those local settings for which $(\tilde{\pi}_K)_\gamma$ is finite for every $\gamma \in \tilde{\pi}_X$. Let W be a set of Reidemeister representatives. Let ζ be an orbit in $R(\tilde{\phi}, \tilde{\xi})$, with

$$I_W(\zeta) = \left\{(\alpha, \sigma): \alpha \in W,\ \sigma \in \tilde{\xi}(\tilde{\pi}_K)_\alpha,\ [\sigma^{-1}\alpha] = \zeta\right\}.$$

Choose a representative $\beta \in \tilde{\pi}_X$ with $[\beta] = \zeta$. We define $\nu(\zeta)$ to be

$$\nu(\zeta) = |(\tilde{\pi}_K)_\beta| \sum_{(\alpha,\sigma) \in I_W(\zeta)} |\tilde{\xi}(\tilde{\pi}_K)_\alpha|^{-1} \in \mathbb{Q}.$$

Note that, for any $\gamma \in \zeta$, $(\tilde{\pi}_K)_\beta$ and $(\tilde{\pi}_K)_\gamma$ are conjugate subgroups of $\tilde{\pi}_K$. Thus $\nu(\zeta)$ is independent of the choice of β.

For any $\alpha \in W$, we define the function $T_\alpha^R : \tilde{\xi}(\tilde{\pi}_K)_\alpha \to \mathbb{Q}(R(\tilde{\phi}, \tilde{\xi}))$ to be, for any $\sigma \in \tilde{\xi}(\tilde{\pi}_K)_\alpha$,

$$T_\alpha^R(\sigma) = \nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha].$$

We may extend T_α^R linearly to $T_\alpha^R : \mathbb{Q}[\tilde{\xi}(\tilde{\pi}_K)_\alpha] \to \mathbb{Q}(R(\tilde{\phi}, \tilde{\xi}))$.

Let Δ_α be a set of right coset representatives for cosets of $(\tilde{\pi}_K)_\alpha$ in $\tilde{\pi}_K$. Let

$$D_q^\alpha = \left\{\delta b: \delta \in \Delta_\alpha,\ b \in B_q\right\}.$$

Then D_q^α is a $\mathbb{Q}[(\tilde{\pi}_K)_\alpha]$-basis for $C_q(\tilde{K}'; \mathbb{Q})$, and the set $\tilde{\imath}(D_q^\alpha)$ is a $\mathbb{Q}[\tilde{\xi}(\tilde{\pi}_K)_\alpha]$-basis for $C_q(\tilde{\imath}(\tilde{K}'); \mathbb{Q})$.

For each $d \in D_q^\alpha$, we have

$$\alpha\tilde{G}_q(d) = \sum_{s \in \tilde{\imath}(\tilde{K}')_q} \mu_{d,\alpha^{-1}s} s. \tag{1}$$

The trace-like sum of $\alpha\tilde{G}_q$ over $\mathbb{Q}[\tilde{\xi}(\tilde{\pi}_K)_\alpha]$ is $\sum_{d \in D_q^\alpha} \sum_{\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha} \mu_{d,\alpha^{-1}\tau\tilde{\imath}(d)}\tau$.

We define $T_\alpha^R(\alpha\tilde{G}_q)$ to be T_α^R applied to the above trace-like sum. Thus

$$T_\alpha^R\left(\alpha\tilde{G}_q\right) = \sum_{d \in D_q^\alpha} \sum_{\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha} \mu_{d,\alpha^{-1}\tau\tilde{\imath}(d)} T_\alpha^R(\tau).$$

Then

$$T_\alpha^R\left(\alpha\tilde{G}_q\right) = \sum_{d \in D_q^\alpha} \sum_{\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha} \mu_{d,\alpha^{-1}\tau\tilde{\imath}(d)} \nu^{-1}([\tau^{-1}\alpha])[\tau^{-1}\alpha]. \tag{2}$$

We use the fact that $\tilde{f}$ is compact to guarantee that the image of $\alpha\tilde{G}_q$ is finitely generated over $\mathbb{Q}[\tilde{\xi}(\tilde{\pi}_K)_\alpha]$, and thus the above sums have at most finitely many nonzero terms. We require all isotropy subgroups for the Reidemeister action to be finite in order to guarantee that ν is defined for every orbit in $R(\tilde{\phi}, \tilde{\xi})$.

Theorem 3.2. *Let* $\mathrm{LS}(f)$ *be a local setting for* f *with* $\tilde{f}$ *compact. Let* W *be a set of Reidemeister representatives. If the isotropy subgroup* $(\tilde{\pi}_K)_\gamma$ *is finite for every* $\gamma \in \tilde{\pi}_X$, *we have*

$$\begin{aligned} L_H\left(f;\, \tilde{f}, \tilde{\imath}\right) &= \sum_q (-1)^q \sum_{\alpha \in W} T_\alpha^R\left(\alpha\tilde{G}_q\right) \\ &= A_H\left(f;\, \tilde{f}, \tilde{\imath}\right) \\ &= N_H\left(f;\, \tilde{f}, \tilde{\imath}\right) \in \mathbb{Z}\left(R\left(\tilde{\phi}, \tilde{\xi}\right)\right). \end{aligned}$$

Thus whenever $L_H(f;\, \tilde{f}, \tilde{\imath})$ *is written in reduced form (with each Reidemeister orbit appearing at most once) the local H-Nielsen number* $N_H(f)$ *is the number of terms with coefficient different from zero.*

Proof. First we prove that $L_h(f;\, \tilde{f}, \tilde{\imath}) = \sum_q (-1)^q \sum_{\alpha \in W} T_\alpha^R(\alpha\tilde{G}_q)$. It suffices to prove that

$$\sum_{\alpha \in W} T_\alpha^R\left(\alpha\tilde{G}_q\right) = \sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \sum_{r \in \mathscr{O}(b)} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta].$$

For any $a \in \tilde{\imath}(\tilde{K}_q')$, let $\rho_{b,a}^\theta = \mu_{b,\theta^{-1}a}$. Then

$$\sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \sum_{r \in \mathscr{O}(b)} \lambda_{r, \theta^{-1}\tilde{\imath}(b)}[\theta] = \sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \rho_{b,\tilde{\imath}(b)}^\theta[\theta].$$

For any $\gamma \in W$, Let Δ_γ be a set of right coset representatives for $(\tilde{\pi}_K)_\gamma$ in $\tilde{\pi}_K$. Then Δ_γ^{-1}, the set of all inverses of elements in Δ_γ, is a set of left coset representatives for $(\tilde{\pi}_K)_\gamma$ in $\tilde{\pi}_K$. Each $x \in \tilde{\pi}_K$ can be expressed uniquely as $x = mn$ with $m \in \Delta_\gamma^{-1}$ and $n \in (\tilde{\pi}_K)_\gamma$. Then $\tilde{\xi}(x)\gamma\tilde{\phi}(x^{-1}) = \tilde{\xi}(m)\gamma\tilde{\phi}(m^{-1})$.

Step 1. We prove that, for any $b \in B_q$, we have

$$\sum_{\theta \in \tilde{\pi}_X} \rho_{b,\tilde{\imath}(b)}^\theta[\theta] = \sum_{\alpha \in W} \sum_{(\gamma,\beta) \in I_W[\alpha]} \sum_{m \in \Delta_\gamma^{-1}} \rho_{b,\tilde{\imath}(b)}^{\tilde{\xi}(m)\beta^{-1}\gamma\tilde{\phi}(m^{-1})} \nu^{-1}([\alpha])[\alpha].$$

Proof of Step 1. We have

$$\begin{aligned} \sum_{\theta \in \tilde{\pi}_X} \rho_{b,\tilde{\imath}(b)}^\theta[\theta] &= \sum_{\alpha \in W} \sum_{\eta \in \Delta_\alpha^{-1}} \rho_{b,\tilde{\imath}(b)}^{\tilde{\xi}(\eta)\alpha\tilde{\phi}(\eta^{-1})}[\alpha] \\ &= \sum_{\alpha \in W} \sum_{x \in \tilde{\pi}_K} \rho_{b,\tilde{\imath}(b)}^{\tilde{\xi}(x)\alpha\tilde{\phi}(x^{-1})} \left|(\tilde{\pi}_K)_\alpha\right|^{-1}[\alpha] \\ &= \sum_{\alpha \in W} \left(\sum_{(\gamma,\delta) \in I_W[\alpha]} \left|\tilde{\xi}(\tilde{\pi}_K)_\gamma\right|^{-1} \right) \sum_{x \in \tilde{\pi}_K} \rho_{b,\tilde{\imath}(b)}^{\tilde{\xi}(x)\alpha\tilde{\phi}(x^{-1})} \nu^{-1}([\alpha])[\alpha]. \end{aligned}$$

For any $(\gamma, \delta) \in I_W[\alpha]$ and for any $x \in \tilde{\pi}_K$, there exists a $z \in \tilde{\pi}_K$ such that

$$\tilde{\xi}(z)\delta^{-1}\gamma\tilde{\phi}(z) = \tilde{\xi}(x)\alpha\tilde{\phi}(x^{-1}).$$

There exist a unique $m \in \Delta_\gamma^{-1}$ and a unique $n \in (\tilde{\pi}_K)_\gamma$ such that $z = mn$. Then

$$\begin{aligned}\tilde{\xi}(x)\alpha\tilde{\phi}(x^{-1}) &= \tilde{\xi}(m)\tilde{\xi}(n)\delta^{-1}\gamma\tilde{\phi}(n^{-1})\tilde{\phi}(m^{-1})\\ &= \tilde{\xi}(m)\tilde{\xi}(n)\delta^{-1}\tilde{\xi}(n^{-1})\gamma\tilde{\phi}(m^{-1}).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{\theta \in \tilde{\pi}_X} \rho^{\theta}_{b,\bar{\iota}(b)}[\theta]\\ &\quad = \sum_{\alpha \in W} \sum_{(\gamma,\delta)\in I_W[\alpha]} |\tilde{\xi}(\tilde{\pi}_K)_\gamma|^{-1}\\ &\qquad \left(\sum_{m \in \Delta_\gamma^{-1}} \sum_{n \in (\tilde{\pi}_K)_\gamma} \rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\tilde{\xi}(n)\delta^{-1}\tilde{\xi}(n^{-1})\gamma\tilde{\phi}(m^{-1})} \nu^{-1}([\alpha])[\alpha] \right).\end{aligned}$$

Let $\langle\delta\rangle = \{y\delta y^{-1}\colon y \in \tilde{\xi}(\tilde{\pi}_K)_\gamma\}$, and let $\omega_\delta = |\tilde{\xi}(\tilde{\pi}_K)_\gamma| / |\langle\delta\rangle|$. Then

$$\omega_\delta = \left|\left\{ y \in \tilde{\xi}(\tilde{\pi}_K)_\gamma \colon y\delta y^{-1} = \delta \right\}\right|.$$

We have

$$\begin{aligned}&\sum_{\theta \in \tilde{\pi}_X} \rho^{\theta}_{b,\bar{\iota}(b)}[\theta]\\ &\quad = \sum_{\alpha \in W} \sum_{(\gamma,\delta)\in I_W[\alpha]} \frac{\omega_\delta}{|\tilde{\xi}(\tilde{\pi}_K)_\gamma|} \sum_{\sigma \in \langle\delta\rangle} \sum_{m \in \Delta_\gamma^{-1}} \rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\sigma^{-1}\gamma\tilde{\phi}(m^{-1})} \nu^{-1}([\alpha])[\alpha]\\ &\quad = \sum_{\alpha \in W} \sum_{(\gamma,\delta)\in I_W[\alpha]} |\langle\delta\rangle|^{-1} \sum_{\sigma \in \langle\delta\rangle} \sum_{m \in \Delta_\gamma^{-1}} \rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\sigma^{-1}\gamma\tilde{\phi}(m^{-1})} \nu^{-1}([\alpha])[\alpha]. \qquad (3)\end{aligned}$$

Note that whenever $\sigma \in \langle\delta\rangle$, we have $(\gamma, \sigma) \in I_W[\alpha]$. For any $(\gamma, \beta) \in I_W[\alpha]$, the sum $\sum_{m \in \Delta_\gamma^{-1}} \rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\beta^{-1}\gamma\tilde{\phi}(m^{-1})}$ appears $|\langle\beta\rangle|$ times in equation (3). Thus equation (3) implies that

$$\sum_{\theta \in \tilde{\pi}_X} \rho^{\theta}_{b,\bar{\iota}(b)}[\theta] = \sum_{\alpha \in W} \sum_{(\gamma,\beta)\in I_W[\alpha]} \sum_{m \in \Delta_\gamma^{-1}} \rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\beta^{-1}\gamma\tilde{\phi}(m^{-1})} \nu^{-1}([\alpha])[\alpha],$$

and Step 1 is proven.

Step 2. We prove that

$$\sum_{\alpha \in W} T^R_\alpha\left(\alpha\tilde{G}_q\right) = \sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \rho^{\theta}_{b,\bar{\iota}(b)}[\theta].$$

Proof of Step 2. By Step 1,

$$\sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \rho^{\theta}_{b,\bar{\iota}(b)}[\theta] = \sum_{b \in B_q} \sum_{\alpha \in W} \sum_{(\gamma,\beta)\in I_W[\alpha]} \sum_{m \in \Delta_\gamma^{-1}} \rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\beta^{-1}\gamma\tilde{\phi}(m^{-1})} \nu^{-1}([\alpha])[\alpha].$$

Note that

$$\rho_{b,\bar{\iota}(b)}^{\tilde{\xi}(m)\beta^{-1}\gamma\tilde{\phi}(m^{-1})} = \rho^1_{b,\tilde{\phi}(m)\gamma^{-1}\beta\tilde{\xi}(m^{-1})\bar{\iota}(b)} = \rho^1_{m^{-1}b,\gamma^{-1}\beta\tilde{\xi}(m^{-1})\bar{\iota}(b)} = \rho^{\beta^{-1}\gamma}_{m^{-1}b,\bar{\iota}(m^{-1}b)}.$$

Recall that, for any $\gamma \in W$, the set $D_q^\gamma = \{ab\colon b \in B_q,\ a \in \Delta_\gamma\}$ is a $\mathbb{Z}[(\tilde{\pi}_K)_\gamma]$-basis for $C_q(\tilde{K}'; \mathbb{Z})$. Let $d = m^{-1}b$. We have

$$\sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \rho^\theta_{b,\tilde{\iota}(b)}[\theta] = \sum_{\alpha \in W} \sum_{(\gamma,\beta) \in I_W[\alpha]} \sum_{d \in D_q^\gamma} \rho^{\beta^{-1}\gamma}_{d,\tilde{\iota}(d)} \nu^{-1}([\alpha])[\alpha].$$

Thus

$$\begin{aligned} \sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \rho^\theta_{b,\tilde{\iota}(b)}[\theta] &= \sum_{\alpha \in W} \sum_{(\gamma,\beta) \in I_W[\alpha]} \sum_{d \in D_q^\gamma} \rho^{\beta^{-1}\gamma}_{d,\tilde{\iota}(d)} \nu^{-1}\left(\left[\beta^{-1}\gamma\right]\right)\left[\beta^{-1}\gamma\right] \\ &= \sum_{\gamma \in W} \sum_{\beta \in (\tilde{\pi}_K)_\gamma} \sum_{d \in D_q^\gamma} \rho^{\beta^{-1}\gamma}_{d,\tilde{\iota}(d)} \nu^{-1}\left(\left[\beta^{-1}\gamma\right]\right)\left[\beta^{-1}\gamma\right]. \end{aligned}$$

By equation (2), we have

$$\sum_{b \in B_q} \sum_{\theta \in \tilde{\pi}_X} \rho^\theta_{b,\tilde{\iota}(b)}[\theta] = \sum_{\gamma \in W} T^R_\gamma\left(\gamma \tilde{G}_q\right) = \sum_{\alpha \in W} T^R_\alpha\left(\alpha \tilde{G}_q\right).$$

Step 2 is proven.

Thus

$$L_H\left(f;\, \tilde{f}, \tilde{\iota}\right) = \sum_q (-1)^q \sum_{\alpha \in W} T^R_\alpha\left(\alpha \tilde{G}_q\right).$$

Next we prove that

$$L_H\left(f;\, \tilde{f}, \tilde{\iota}\right) = A_H\left(f;\, \tilde{f}, \tilde{\iota}\right) = N_H\left(f;\, \tilde{f}, \tilde{\iota}\right).$$

Consider one element $\alpha \in W$. For each $\sigma \in \tilde{\xi}(\tilde{\pi}_K)_\alpha$, we find a connection between $T^R_\alpha(\alpha \tilde{G}_q)$ and $T(\sigma^{-1}\alpha \tilde{G}_q) = \sum_{x \in \tilde{K}'_q} \mu_{x,\alpha^{-1}\sigma\tilde{\iota}(x)}$.

For any $c \in \tilde{K}'_q$ and any $\sigma \in \tilde{\xi}(\tilde{\pi}_K)_\alpha$, we have $\sigma^{-1}\alpha \tilde{G}_q(c) = \sum_{a \in \tilde{\iota}(\tilde{K}'_q)} \mu_{c,\alpha^{-1}\sigma a} a$, and $T(\sigma^{-1}\alpha \tilde{G}_q) = \sum_{c \in \tilde{K}'_q} \mu_{c,\alpha^{-1}\sigma\tilde{\iota}(c)}$.

For some $\eta \in (\tilde{\pi}_K)_\alpha$ and for some $d \in D_q^\alpha$, we have $c = \eta d$. Thus, by Proposition 3.1 and equation (1),

$$\begin{aligned} \sigma^{-1}\alpha \tilde{G}_q(c) &= \sigma^{-1}\alpha \tilde{G}_q(\eta d) = \sigma^{-1}\tilde{\xi}(\eta)\alpha \tilde{G}_q(d) \\ &= \sigma^{-1}\tilde{\xi}(\eta) \sum_{x \in D_q^\alpha} \sum_{\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha} \rho^\alpha_{d,\tau\tilde{\iota}(x)} \tau \tilde{\iota}(x) \\ &= \sigma^{-1}\tilde{\xi}(\eta)\left(\sum_{\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha} \rho^\alpha_{d,\tau\tilde{\iota}(d)} \tau\tilde{\iota}(d) + \sum_{x \neq d} \sum_{\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha} \rho^\alpha_{d,\tau\tilde{\iota}(x)} \tau\tilde{\iota}(x) \right). \end{aligned}$$

To find the coefficient of $\tilde{\iota}(c) = \tilde{\iota}(\eta d) = \tilde{\xi}(\eta)\tilde{\iota}(d)$, we must find the $\tau \in \tilde{\xi}(\tilde{\pi}_K)_\alpha$ for which $\sigma^{-1}\tilde{\xi}(\eta)\tau = \tilde{\xi}(\eta)$. Thus we need $\tau = \tilde{\xi}(\eta^{-1})\sigma\tilde{\xi}(\eta)$. The contribution of c to $T(\sigma^{-1}\alpha \tilde{G}_q)$ is $\rho^\alpha_{d,\tilde{\xi}(\eta^{-1})\sigma\tilde{\xi}(\eta)\tilde{\iota}(d)}$. Thus

$$T\left(\sigma^{-1}\alpha \tilde{G}_q\right) = \sum_{\eta \in (\tilde{\pi}_K)_\alpha} \sum_{d \in D_q^\alpha} \rho^\alpha_{d,\tilde{\xi}(\eta^{-1})\sigma\tilde{\xi}(\eta)\tilde{\iota}(d)} \in \mathbb{Z}.$$

As before, let $\langle\sigma\rangle = \{\theta^{-1}\sigma\theta\colon \theta \in \tilde{\xi}(\tilde{\pi}_K)_\alpha\}$ be the orbit of σ under the action of conjugation in $\tilde{\xi}(\tilde{\pi}_K)_\alpha$, and let $C(\alpha)$ be the set of conjugacy classes in $\tilde{\xi}(\tilde{\pi}_K)_\alpha$. Let

$\omega_\sigma = |\{\theta \in \tilde{\xi}(\tilde{\pi}_K)_\alpha\colon \theta^{-1}\sigma\theta = \sigma\}|$ be the size of the isotropy subgroup for σ under this action. Then we have

$$\omega_\sigma = \frac{|\tilde{\xi}(\tilde{\pi}_K)_\alpha|}{|\langle\sigma\rangle|}.$$

We may write $T(\sigma^{-1}\alpha\tilde{G}_q)$ as $T(\sigma^{-1}\alpha\tilde{G}_q) = \omega_\sigma \Sigma_{\beta\in\langle\sigma\rangle}\Sigma_{d\in D_q^\alpha}\rho^\alpha_{d,\beta\tilde{\imath}(d)}$. Note that for $\beta \in \langle\sigma\rangle$, $T(\sigma^{-1}\alpha\tilde{G}_q) = T(\beta^{-1}\alpha\tilde{G}_q)$. We compare this sum with $T^R_\alpha(\alpha\tilde{G}_q)$ as in equation (2).

Note that if $\beta \in \langle\sigma\rangle$, then $[\sigma^{-1}\alpha] = [\beta^{-1}\alpha]$. We may write $T^R_\alpha(\alpha\tilde{G}_q)$ as

$$T^R_\alpha\left(\alpha\tilde{G}_q\right) = \sum_{\langle\sigma\rangle\in C(\alpha)}\ \sum_{d\in D_q^\alpha}\ \sum_{\beta\in\langle\sigma\rangle} \rho^\alpha_{d,\beta\tilde{\imath}(d)}\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha]. \tag{4}$$

Then

$$\begin{aligned} T^R_\alpha\left(\alpha\tilde{G}_q\right) &= \sum_{\langle\sigma\rangle\in C(\alpha)}\left(\frac{1}{\omega_\sigma}T\left(\sigma^{-1}\alpha\tilde{G}_q\right)\right)\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha] \\ &= \sum_{\sigma\in\tilde{\xi}(\tilde{\pi}_K)_\alpha}\frac{1}{|\langle\sigma\rangle|}\frac{1}{\omega_\sigma}T\left(\sigma^{-1}\alpha\tilde{G}_q\right)\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha] \\ &= \sum_{\sigma\in\tilde{\xi}(\tilde{\pi}_K)_\alpha}\frac{1}{|\tilde{\xi}(\tilde{\pi}_K)_\alpha|}T\left(\sigma^{-1}\alpha\tilde{G}_q\right)\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha]. \end{aligned}$$

Let $L_\alpha(f;\tilde{f})$ be the sum $L_\alpha(f;\tilde{f}) = \Sigma_q(-1)^q T^R_\alpha(\alpha\tilde{G}_q)$, and note that $L_H(f;\tilde{f},\tilde{\imath}) = \Sigma_{\alpha\in W}L_\alpha(f;\tilde{f})$. Then

$$\begin{aligned} L_\alpha(f;\tilde{f}) &= \sum_{\sigma\in\tilde{\xi}(\tilde{\pi}_K)_\alpha}\frac{1}{|\tilde{\xi}(\tilde{\pi}_K)_\alpha|}\sum_q(-1)^q T\left(\sigma^{-1}\alpha\tilde{G}_q\right)\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha] \\ &= \sum_{\sigma\in\tilde{\xi}(\tilde{\pi}_K)_\alpha}\frac{\Lambda\left(\sigma^{-1}\alpha\tilde{f}\right)}{|\tilde{\xi}(\tilde{\pi}_K)_\alpha|}\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha]. \end{aligned}$$

We have

$$L_H\left(f;\tilde{f},\tilde{\imath}\right) = \sum_{\alpha\in W}\ \sum_{\sigma\in\tilde{\xi}(\tilde{\pi}_K)_\alpha}\frac{\Lambda\left(\sigma^{-1}\alpha\tilde{f}\right)}{|\tilde{\xi}(\tilde{\pi}_K)_\alpha|}\nu^{-1}([\sigma^{-1}\alpha])[\sigma^{-1}\alpha].$$

Next we combine like Reidemeister orbits. Let $\alpha \in W$. For any $(\beta, \delta) \in I_W([\alpha])$, $\Lambda(\delta^{-1}\beta\tilde{f}) = \Lambda(\alpha\tilde{f})$ by Proposition 2.11. Of course, we have $\nu^{-1}([\delta^{-1}\beta]) = \nu^{-1}([\alpha])$.

The local generalized H-Lefschetz number reduces to

$$L_H\left(f;\tilde{f},\tilde{\imath}\right) = \sum_{\alpha\in W}\left(\sum_{(\beta,\delta)\in I_W([\alpha])}\frac{1}{|\tilde{\xi}(\tilde{\pi}_K)_\beta|}\right)\Lambda\left(\alpha\tilde{f}\right)\nu^{-1}([\alpha])[\alpha].$$

Recall that

$$\nu([\alpha]) = |(\tilde{\pi}_K)_\alpha|\sum_{(\beta,\delta)\in I_W([\alpha])}\frac{1}{|\tilde{\xi}(\tilde{\pi}_K)_\beta|}.$$

Thus

$$L_H(f;\tilde{f},\tilde{\imath}) = \sum_{\alpha\in W} \frac{\Lambda(\alpha\tilde{f})}{|(\tilde{\pi}_K)_\alpha|}[\alpha] = A_H(f;\tilde{f},\tilde{\imath}).$$

By Proposition 2.10, we have $L_H(f;\tilde{f},\tilde{\imath}) = \sum_{\alpha\in W} i(N_H^\alpha)[\alpha] = N_H(f;\tilde{f},\tilde{\imath})$. □

3.2. Local weak Jiang settings

3.2.1. The Jiang subgroup

Given a map $f: X \to X$ with Fix f compact, the Jiang subgroup $T(f, x_0)$ of $\pi_1(X, f(x_0))$ is defined as follows. Given $\alpha \in \pi_1(X, f(x_0))$, we have $\alpha \in T(f, x_0)$ if and only if there exists a homotopy $h: X\times I \to X$ with $h_0 = f = h_1$ so that the loop $h(x_0, \cdot)$ is an element of the loop class α. If $T(f, x_0) = \pi_1(X, f(x_0))$, then all Nielsen classes for f have the same index. Thus if the Lefschetz number for f is zero, the Nielsen number for f is also zero. The space X is a Jiang space if $T(\mathrm{id}_X, x_0) = \pi_1(X, x_0)$. For a Jiang space, the Jiang subgroup of any map f is the entire fundamental group of X. A Jiang space has the property that all lifts of a map to the universal cover of the space are homotopic via a homotopy that preserves fibers. (See [1,9].)

Definition 3.3 (*Local weak Jiang setting*). A local setting LS(f) for f is a local weak Jiang setting if for every $\alpha \in \tilde{\pi}_X$, $\alpha\tilde{f}$ is freely homotopic to $\tilde{f}$ via an admissible homotopy J. That is, for each $t \in I$, we must have $\mathrm{Coin}(J_t, \tilde{\imath}) \cap \partial\tilde{K} = \emptyset$.

This is the local version of the weak Jiang condition defined in [2]. If the local Lefschetz number $\lambda(f)$ is zero in a local weak Jiang setting, we have the local H-Nielsen number $N_H(f)$ also equal to zero.

Corollary 3.4. *Let* LS(f) *satisfy the hypotheses of Theorem* 3.2, *and let* LS(f) *be a local weak Jiang setting for f. Then*

$$L_H(f;\tilde{f},\tilde{\imath}) = A_H(f;\tilde{f},\tilde{\imath}) = \Lambda(\tilde{f}) \sum_{\alpha\in W} |(\tilde{\pi}_K)_\alpha|^{-1}[\alpha] = N_H(f;\tilde{f},\tilde{\imath}).$$

If $\tilde{\pi}_K$ and $\tilde{\pi}_X$ are finite, we have

$$L_H(f;\tilde{f},\tilde{\imath}) = A_H(f;\tilde{f},\tilde{\imath}) = \frac{\lambda(f)}{|\tilde{\pi}_X|} \sum_{\alpha\in W} |[\alpha]|[\alpha] = N_H(f;\tilde{f},\tilde{\imath})$$

with $\lambda(f)$ the local Lefschetz number of f as in [6]. *Thus for $\tilde{\pi}_K$ and $\tilde{\pi}_X$ finite, whenever $\lambda(f) = 0$, we have $N_H(f) = 0$.*

Proof. If LS(f) is a local weak Jiang setting, then for all $\alpha \in \tilde{\pi}_X$, $\Lambda(\alpha\tilde{f}) = \Lambda(\tilde{f})$. If $|\tilde{\pi}_K| < \infty$, we have $|(\tilde{\pi}_K)_\alpha|^{-1} = |\tilde{\pi}_K|^{-1}|[\alpha]|$. The local Lefschetz number $\lambda(f)$ is the sum of the indices of the local H-Nielsen classes of f. Thus

$$\lambda(f) = \Lambda(\tilde{f}) \sum_{\alpha \in W} |(\tilde{\pi}_K)_\alpha|^{-1} = \frac{\Lambda(\tilde{f})}{|(\tilde{\pi}_K)|} \sum_{\alpha \in W} |[\alpha]|$$
$$= \frac{\Lambda(\tilde{f})|(\tilde{\pi}_X)|}{|(\tilde{\pi}_K)|}. \qquad \square$$

3.3. Sufficient conditions for $(\tilde{\pi}_K)_\alpha$ to be finite

The preceding theorem requires that $\tilde{f}$ be compact and that the isotropy subgroup $(\tilde{\pi}_K)_\alpha$ be finite for every $\alpha \in W$. We consider the restrictions on LS(f) imposed by these requirements.

If $\tilde{\pi}_K$ is a finite group, we have $\tilde{f}$ compact and all isotropy subgroups finite. Thus we may apply the theorem to any setting LS(f) with $\tilde{\pi}_K$ finite.

If $\tilde{\pi}_K$ is an infinite group and $\tilde{f}$ is compact, the situation is more complicated. As in [2], the map $\tilde{f}$ is compact if and only if the image of $\tilde{\phi}$ is finite. For the case in which $K = L$, if the image of $\tilde{\phi}$ is finite, then every isotropy subgroup is finite. In the local case ($K \neq L$), this is not true.

Proposition 3.5. *Let* LS(f) *be a setting for f. If the image of $\tilde{\phi}$ is finite and if for every $\alpha \in \tilde{\pi}_X$ we have* $\ker \tilde{\xi} \cap (\tilde{\pi}_K)_\alpha$ *finite, all isotropy subgroups are finite.*

Proof. For each $\sigma \in (\tilde{\pi}_K)_\alpha$, we have $\tilde{\xi}(\sigma)\alpha\tilde{\phi}(\sigma^{-1}) = \alpha$. Thus $\tilde{\xi}(\sigma) = \alpha\tilde{\phi}(\sigma)\alpha^{-1}$. Because im $\tilde{\phi}$ is finite, there are at most finitely many elements in $\tilde{\xi}(\tilde{\pi}_K)_\alpha$. We have

$$\frac{(\tilde{\pi}_K)_\alpha}{\left(\ker \tilde{\xi} \cap (\tilde{\pi}_K)_\alpha\right)} \cong \tilde{\xi}(\tilde{\pi}_K)_\alpha.$$

Thus for $\ker \tilde{\xi} \cap (\tilde{\pi}_K)_\alpha$ finite and $\tilde{\xi}(\tilde{\pi}_K)_\alpha$ finite, we have $(\tilde{\pi}_K)_\alpha$ finite. $\square$

Proposition 3.6. *Let* LS(f) *be a setting for f with $\tilde{\pi}_K$ an infinite group and $(\tilde{\pi}_K)_\alpha$ finite for all $\alpha \in \tilde{\pi}_X$. Then $\tilde{\pi}_X$ is infinite.*

Proof. We have $|\tilde{\pi}_K : (\tilde{\pi}_K)_\alpha| = |[\alpha]|$ for all $\alpha \in \tilde{\pi}_X$. Also, $|\tilde{\pi}_X| = \sum_{\alpha \in W} |[\alpha]|$. If $\tilde{\pi}_X$ is finite, each $[\alpha]$ is finite. Then the fact that $(\tilde{\pi}_K)_\alpha$ is finite forces $\tilde{\pi}_K$ to be finite. This contradicts the hypotheses. Thus $\tilde{\pi}_X$ is infinite. $\square$

3.3.1. Conclusion

Theorem 3.2 applies to a local setting for f if $\tilde{f}$ is compact and

(1) $\tilde{\pi}_K$ is finite or

(2) $\tilde{\pi}_K$ is infinite, im $\tilde{\phi}$ is finite and $\ker \tilde{\xi} \cap (\tilde{\pi}_K)_\alpha$ is finite for all $\alpha \in \tilde{\pi}_X$.

By Proposition 3.6, if $\tilde{\pi}_X$ is a finite group, we must have $\tilde{\pi}_K$ finite in order to apply Theorem 3.2.

4. Examples

First we provide an example of a space that is not a Jiang space, but for which each self-map gives rise to a weak Jiang setting. See [2]. We calculate the Nielsen number (with $U = X$ and $H = 1$) of any self-map on this space using Corollary 3.4.

Second, we calculate local Nielsen numbers (with $H = 1$) of maps $f: K \to L$, where L is a lens space and K is a solid torus in L. This provides an example of a local setting in which it is necessary to choose a $(1, f)$-admissible cover for K that is not a subspace of $\tilde{L}$.

4.1. The Nielsen number of a self-map on a spherical space form

Let M be the orbit space for an orthogonal, free action of the dihedral group of order 12 on S^3. Then M is a spherical space form. (See [12].) Let the dihedral group of order 12 be denoted by $D_{12} = \langle X, Y\colon X^4 = I, Y^3 = I, XY = Y^2X \rangle$, with Y and X as below.

$$Y = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ 0 & 0 & \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Any element of D_{12} can be written uniquely in the form Y^iX^j with $0 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 3$. We have $Y^mX^nY^iX^j = Y^rX^s$ with $r \equiv m + 2^n i \bmod 3$ and $s \equiv n + j \bmod 4$. Note that $XX^{\mathrm{T}} = I$ and $YY^{\mathrm{T}} = I$. Thus any element of D_{12} acts orthogonally on S^3.

Proposition 4.1. *The given action of D_{12} on S^3 is free.*

Proof. Let $Z \in D_{12}$ with $Z \neq I$. If x is a fixed point of the action of the matrix Z on S^3, then $(Z - I)x = 0$, and 1 is an eigenvalue of Z with eigenvector x. We prove that, for each $Z \neq I$, $\det(Z - I) \neq 0$. Thus 1 is not an eigenvalue of Z. Note that for any 4×4 matrix A, we have $\det(-A) = \det(A)$. Because $X^2 = -I$, we have $\det(X^2 - I) = \det(-2I) = 2^4 \neq 0$. Also, $\det(X^2 - I) = \det(X - I)\det(X + I) = \det(X - I)\det(X^3 - I)$. Thus X, X^2 and X^3 act on S^3 without fixed points.

Consider $\det(Y - I) = 9 \neq 0$. We have $\det(Y - I) = \det(Y^4 - I) = \det(Y - I)\det(Y + I)\det(Y^2 + I) = \det(Y - I)\det(YX^2 - I)\det(Y^2X^2 - I)$. Also, $\det(Y^2X^2 - I) = \det(YX - I)\det(YX^3 - I)$, and $\det(YX^2 - I) = \det(Y^4X^2 - I) = \det(Y^2X - I)\det(Y^2X^3 - I)$. Thus none of these determinants is zero, and all nontrivial elements of D_{12} act on S^3 without fixed points. $\square$

Note that $S^3/\langle Y\rangle$ is the lens space $L(3, 1)$. Thus we have $S^3 \to L(3, 1) \to M$ a composition of covering transformations.

Proposition 4.2. *For every map $f: M \to M$, any lift $\hat{f}$ of f to S^3 gives rise to a setting* $\mathrm{LS}(f)$ *for f (with $U = M$ and $H = 1$) that is a weak Jiang setting. However, M is not a Jiang space.*

Proof. A Jiang space has an Abelian fundamental group. Thus M is not a Jiang space.

A weak Jiang setting for f is a setting as in Definition 2.1 with $K = L$, $\tilde{K} = \tilde{L}$ and for which all lifts of f to the covering space are freely homotopic. It remains to prove that for every $Z \in D_{12}$ we have $\deg(Z\hat{f}) = \deg(\hat{f})$. For $Z = I$, we have $\deg(Z\hat{f}) = \deg(Z)\deg(\hat{f})$, and the covering transformation Z has no fixed points on S^3. We have $L(Z) = 0 = 1 - \deg(Z)$. Thus $\deg(Z) = 1$. □

Let f and $\hat{f}$ be as in Proposition 4.2. As in the proof of Corollary 3.4, $L(\hat{f}) = L(f)$. Because $\mathrm{LS}(f)$ is a weak Jiang setting, we have $N(f) = 0$ whenever $L(f) = 0$.

Let $\hat{\phi}: D_{12} \to D_{12}$ be the endomorphism determined by $\hat{f}Z = \hat{\phi}(Z)\hat{f}$ for any $Z \in D_{12}$. By Corollary 3.4 with $U = X$ and $H = 1$ (see also [2]),

$$L_1(f; \hat{f}, \hat{\imath}) = L(f) \sum_{\alpha \in W} |(D_{12})_\alpha|^{-1}[\alpha] \in \mathbb{Z}\big(R(\hat{\phi}, \hat{\xi})\big)$$

with $L(f)$ the standard Lefschetz number of f. Note that here $\hat{\xi}$ is the identity function.

Proposition 4.3. *For each endomorphism ψ of D_{12}, there is a self-map $g: M \to M$ that fixes the base point $x_0 \in M$ with $\psi = g_\#: D_{12} \to D_{12}$.*

Proof. Let g be defined on the 1-skeleton of M with $\psi = g_\#$. Then we may extend g to the 2-skeleton of M. Note that $\pi_2(M) = \pi_2(S^3) = 0$. Thus for the attaching map h for any 3-cell of M, we have $gh \simeq h$. This implies that we may extend g to M. □

Let $\hat{x}_0$ be the base point of S^3 in the fiber above x_0. Because S^3 is the universal cover of M, there is a lift $\hat{g}$ of g to S^3 that fixes $\hat{x}_0$. We have $\hat{g}Z = \psi(Z)\hat{g}$, and thus there is a setting for g with $\hat{\phi} = \psi$ to which Corollary 3.4 can be applied. Every endomorphism of D_{12} appears as $\hat{\phi}$ for some setting of some map $f: M \to M$.

Proposition 4.4. *The endomorphisms of D_{12} are of the form $\hat{\phi}(X) = Y^\alpha X^\beta$ and $\hat{\phi}(Y) = Y^\gamma$, with $0 \leqslant \alpha \leqslant 2$, $0 \leqslant \gamma \leqslant 2$, and $0 \leqslant \beta \leqslant 3$ so that whenever β is even, $\alpha = \gamma = 0$.*

Proof. For any endomorphism $\hat{\phi}$, there exists a unique 4-tuple $(\alpha, \beta, \gamma, \delta)$ with $\alpha, \gamma \in \{0, 1, 2\}$ and $\beta, \delta \in \{0, 1, 2, 3\}$ such that $\hat{\phi}(X) = Y^\alpha X^\beta$ and $\hat{\phi}(Y) = Y^\gamma X^\delta$.

Because $\hat{\phi}(X^4) = I$, we have $I = \hat{\phi}(X^4) = (Y^\alpha X^\beta)^4 = Y^{\alpha+2^\beta\alpha+2^{2\beta}\alpha+2^{3\beta}\alpha}X^{4\beta}$. Thus for any β we must have $\alpha(1 + 2^\beta + 2^{2\beta} + 2^{3\beta}) \equiv 0 \bmod 3$. Note that $2^{2\beta} \equiv 1 \bmod 3$, and $2^{3\beta} \equiv 2^\beta \bmod 3$. We have, for any β, $2\alpha(1 + 2^\beta) \equiv 0 \bmod 3$. For β even, this forces α to be 0.

Because $\hat{\phi}(Y^3) = I$, we have $I = \hat{\phi}(Y^3) = (Y^\gamma X^\delta)^3 = Y^{\gamma+2^\delta\gamma+2^{2\delta}\gamma}X^{3\delta}$. We must have $3\delta \equiv 0 \bmod 4$. Thus $\delta = 0$.

Finally, we know $\hat{\phi}(XY) = \hat{\phi}(Y^2X)$. We have $\hat{\phi}(XY) = Y^\alpha X^\beta Y^\gamma = Y^{\alpha+2^\beta\gamma}X^\beta$, and $\hat{\phi}(Y^2X) = Y^{2\gamma+\alpha}X^\beta$. We must have $2\gamma + \alpha \equiv \alpha + 2^\beta\gamma \bmod 3$. Thus $\gamma(2^\beta - 2) \equiv 0 \bmod 3$. For β even, this forces γ to be 0. $\square$

We will determine the Lefschetz number of f and the Nielsen number of f using the data α, β, γ and the degree of $\hat{f}$.

First we compute the Lefschetz number $L(f)$. Because M is orientable, we can use Poincaré duality to find $H_2(M)$. The homology of M is $H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}_4$, $H_2(M) = 0$ and $H_3(M) = \mathbb{Z}$. Thus the Lefschetz number of f equals $L(f) = 1 - \beta - d$, with d equal to the degree of $\hat{f}$.

Next we apply Corollary 3.4 to a setting for f and calculate the Nielsen number of f in terms of $L(f)$. Tedious calculations are required to find the Reidemeister orbits for each possible $\hat{\phi}$.

We provide the details of the calculation for the homomorphism $\hat{\phi}$ given by $(\alpha, \beta, \gamma) = (2, 3, 2)$. The calculations for the other homomorphisms are similar.

We require the following fact. If Y^mX^n acts on Y^iX^j via the Reidemeister action, the result is $Y^\theta X^\tau$ with

$$\theta \equiv m + 2^n i + 2^{n+j+1}(1 + 8 + 8^2 + \cdots + 8^{3-n}) + 2^{9-2n+j}(3 - m) \bmod 3$$

and $\tau \equiv 8 - 2n + j \bmod 4$. The orbit of the identity is

$$[1] = \{Y^\theta X^\tau\colon i = 0,\ j = 0,\ 0 \leqslant m \leqslant 2,\ 0 \leqslant n \leqslant 3\}$$
$$= \{1, Y, Y^2, X^2, YX^2, Y^2X^2\}.$$

Similarly,

$$[X] = \{Y^\theta X^\tau\colon i = 0,\ j = 1,\ 0 \leqslant m \leqslant 2,\ 0 \leqslant n \leqslant 3\} = \{X, Y^2X^3\}.$$

Note that when j is replaced by $j + 2$, θ does not change. Thus $[X^3] = \{X^3, Y^2X\}$. It remains to check that $[YX] = \{YX, YX^3\}$, but X acts on YX via the Reidemeister action with the result YX^3. Thus the distinct orbits are [1], $[X]$, $[X^3]$, and $[YX]$. By Corollary 3.4, we have

$$L_1(f; \hat{f}, \hat{\imath}) = \frac{L(f)}{6}(3[1] + [X] + [X^3] + [YX]).$$

Thus

$$N(f) = \begin{cases} 4 & \text{if } L(f) \neq 0, \\ 0 & \text{if } L(f) = 0. \end{cases}$$

Table 1

(α, β, γ)	$L_1(f; \hat{f}, \hat{\imath})$	$N(f)$
(0, 0, 0)	$L(f)[1]$	1 or 0
(0, 2, 0)	$L(f)[1]$	1 or 0
(0, 3, 0)	$\frac{L(f)}{2}([1]+[X])$	2 or 0
(1, 3, 0)	$\frac{L(f)}{2}([1]+[X])$	2 or 0
(2, 3, 0)	$\frac{L(f)}{2}([1]+[X])$	2 or 0
(0, 1, 0)	$\frac{L(f)}{4}([1]+[X]+[X^2]+[X^3])$	4 or 0
(1, 1, 0)	$\frac{L(f)}{4}([1]+[X]+[X^2]+[X^3])$	4 or 0
(2, 1, 0)	$\frac{L(f)}{4}([1]+[X]+[X^2]+[X^3])$	4 or 0
(0, 3, 1)	$\frac{L(f)}{6}([1]+[Y]+[Y^2]+3[X])$	4 or 0
(1, 3, 1)	$\frac{L(f)}{6}([1]+3[X]+[X^2]+[Y])$	4 or 0
(2, 3, 1)	$\frac{L(f)}{6}([1]+3[X]+[X^2]+[Y^2])$	4 or 0
(0, 3, 2)	$\frac{L(f)}{6}(3[1]+[X]+[Y^2X]+[YX])$	4 or 0
(1, 3, 2)	$\frac{L(f)}{6}(3[1]+[X]+[X^3]+[Y^2X])$	4 or 0
(2, 3, 2)	$\frac{L(f)}{6}(3[1]+[X]+[X^3]+[YX])$	4 or 0
(0, 1, 1)	$\frac{L(f)}{12}([1]+2[Y]+3[X]+[X^2]+3[X^3]+2[YX^2])$	6 or 0
(1, 1, 1)	$\frac{L(f)}{12}(2[1]+[Y]+3[X]+2[X^2]+3[X^3]+[YX^2])$	6 or 0
(2, 1, 1)	$\frac{L(f)}{12}(2[1]+3[X]+[Y^2]+2[X^2]+3[X^3]+[Y^2X^2])$	6 or 0
(0, 1, 2)	$\frac{L(f)}{12}(3[1]+[X]+3[X^2]+[X^3]+2[YX]+2[YX^3])$	6 or 0
(1, 1, 2)	$\frac{L(f)}{12}(3[1]+2[X]+3[X^2]+2[X^3]+[Y^2X]+[Y^2X^3])$	6 or 0
(2, 1, 2)	$\frac{L(f)}{12}(3[1]+2[X]+3[X^2]+2[X^3]+[YX]+[YX^3])$	6 or 0

We replace $\lambda(f)$ in Corollary 3.4 by $L(f)$, the usual Lefschetz number of f, because we are considering a case for which f is defined on all of M.

Recall that for $f: M \to M$, we have $L(f)=0 \Leftrightarrow N(f)=0$. The results for each homomorphism $\hat{\phi}$ are listed in Table 1. Note that $N(f)$ is independent of α, but $L_1(f; \hat{f}, \hat{\imath})$ is not.

Note that $L(f)=0$ if and only if the degree of $\hat{f}$ equals $1-\beta$. By combining this with the information in the table, we determine $N(f)$ completely in terms of β, γ and the degree of $\hat{f}$.

4.2. Local Nielsen numbers for lens spaces

Let p and q be relatively prime integers. Let $\mathbb{Z}_p=\langle\gamma\colon \gamma^p=1\rangle$ act on S^3 by

$$\gamma\cdot(z_1, z_2)=\left(e^{\frac{2\pi}{p}i}z_1, e^{\frac{2\pi}{p}qi}z_2\right).$$

The orbit space is the lens space $L(p, q)$. Let U be an open connected subspace of $L(p, q)$. Let $\tilde{p}_L: S^3\to L(p, q)$ be the covering projection and $f: U\to L(p, q)$ be a map with Fix $f\cap\partial U=\emptyset$. Let L be a simplicial complex for which there exists a stable (f, U)-complex K as in Section 2, with $\tilde{L}=S^3$. For K a solid torus, we will calculate the local generalized Lefschetz number $L_1(f;\tilde{f}, \tilde{\imath})$ and the local Nielsen number $N(f)$ in terms of the lift index $\Lambda(\tilde{f})$ and the homomorphism $\tilde{\phi}$ induced by $\tilde{f}$.

Let LS(f) be a local setting for f. For K a solid torus, consider the normal subgroup J of $\pi_1(K)$ given by $J=\{x^p\colon x\in\pi_1(K)\}$. Let $\tilde{K}$ be a covering space for K such that $\tilde{\pi}_K\cong\pi_1(K)/J$. Then $\tilde{K}$ is an (H, f)-admissible covering space for K with $H=1$ for any $f: K\to L$. We have $\tilde{\pi}_K=\mathbb{Z}_p=\langle\beta\colon \beta^p=1\rangle$. For any homomorphism $\psi:\mathbb{Z}_p\to\mathbb{Z}_p$, we have $\psi(\beta)=\gamma^k$ for some $0\leqslant k\leqslant p-1$.

Let $\tilde{\phi}$ be the homomorphism induced by $\tilde{f}$. We have $\tilde{\imath}:\tilde{K}\to\tilde{L}$ inducing $\tilde{\xi}$. If K is not contractible in L, we have $\tilde{\xi}(\beta)=\gamma$. If K is contractible in L, we have $\tilde{\xi}(\beta)=1$.

Proposition 4.5. *First, consider K a solid torus not contractible in L. If $\tilde{\phi}(\beta)=\gamma$, there are p Reidemeister orbits, and each is a singleton. If $\tilde{\phi}(\beta)=\gamma^s$ and $1<s\leqslant p$, there are m Reidemeister orbits each containing p/m elements where $m=(p, |1-s|)$.*

Second, consider K a solid torus contractible in L. If $\tilde{\phi}(\beta)=\gamma$, there is one Reidemeister orbit. If $\tilde{\phi}(\beta)=\gamma^s$ and $1<s\leqslant p$, there are n Reidemeister orbits each containing p/n elements where $n=(p, s)$.

Proof.

Case 1: *The solid torus K is not contractible in L.* For $s=1$, $[\gamma^i]=\{\tilde{\xi}(\beta^j)\gamma^i\tilde{\phi}(\beta^{-j})\colon 0\leqslant j<p\}=\{\gamma^i\}$ is a singleton for each i.

For $\tilde{\phi}(\beta)=\gamma^s$ and $i<s\leqslant p$, we have $[\gamma^i]=\{\tilde{\xi}(\beta^j)\gamma^i\tilde{\phi}(\beta^{-sj})\colon 0\leqslant j<p\}=\{\gamma^{i+j(1-s)}\colon 0\leqslant j<p\}$ for each i. The smallest integer l such that $l(1-s)\equiv 0 \bmod p$ is $l=p/m$ with $m=(p, |1-s|)$. Thus $[\gamma]=\{\gamma^i, \gamma^{i+(1-s)}, \gamma^{i+2(1-s)},\ldots, \gamma^{i+(p/m-1)(1-s)}\}$, and the elements listed are distinct. Each orbit has p/m elements. Thus the number of classes is m.

Case 2: *The solid torus K is contractible in L.* The calculations are similar to those for Case 1. We see that if $\tilde{\phi}(\beta)=\gamma$, $[\gamma^i]=\tilde{\pi}_L$ for each i. If $\tilde{\phi}(\beta)=\gamma^s$ with $1<s\leqslant p$, we have $[\gamma^i]=\{\gamma^i, \gamma^{i-s}, \gamma^{i-2s},\ldots,\gamma^{i-(p/n-1)s}\}$ for each i. □

The covering transformation $\gamma : S^3 \to S^3$ is homotopic to the identity. Thus we have $\gamma^i \tilde{f} \simeq \tilde{f}$ for all i. The local setting LS(f) is therefore a local weak Jiang setting. We have, by Corollary 3.4,

$$L_1(f; \tilde{f}, \tilde{\imath}) = \frac{\lambda(f)}{p} \sum_{\alpha \in W} |[\alpha]| [\alpha].$$

For a given $\tilde{\phi}$, all of the Reidemeister orbits are the same size. Thus we have

$$L_1(f; \tilde{f}, \tilde{\imath}) = \lambda(f) \, |R(\tilde{\phi})|^{-1} \sum_{\alpha \in W} [\alpha].$$

For a local setting LS(f) with K a solid torus not contractible in L, the local Nielsen number of f is

$$N(f) = \begin{cases} p & \text{for } s = 1 \text{ and } \lambda(f) \neq 0, \\ m & \text{for } s \neq 1,\ m = (p, |1 - s|) \textit{ and } \lambda(f) \neq 0, \\ 0 & \text{for } \lambda(f) = 0. \end{cases}$$

For a local setting LS(f) with K a solid torus contractible in L, the local Nielsen number of f is

$$N(f) = \begin{cases} p & \text{for } s = 1 \text{ and } \lambda(f) \neq 0, \\ n & \text{for } s \neq 1,\ n = (p, s) \text{ and } \lambda(f) \neq 0, \\ 0 & \text{for } \lambda(f) = 0. \end{cases}$$

Note that whenever K is a solid torus not contractible in L we have $\tilde{K}$ a component of $\overline{K}$. In that case, we have a local setting as in [4], where $\tilde{K}$ is required to be a subset of $\tilde{L}$. If K is a solid torus that is contractible in L, we cannot use a local setting as in [4]. To see this we consider such a K and a map $f : K \to L$ for which $f_{\#}(\beta) = \gamma$. The space $\overline{K}$ consists of p disjoint copies of K. If we were forced to choose $\tilde{K}$ a subset of $\tilde{L}$, we would have $\tilde{K}$ homeomorphic to K. There would be no lift of f to $\tilde{K}$, and $\tilde{K}$ would not be a (1, f)-admissible cover for K. Thus the complications introduced in [5] by allowing $\tilde{K}$ not to be a subset of $\tilde{L}$ are necessary for studying local Nielsen fixed point theory using the local generalized H-Lefschetz number.

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